Exploring Piezoelectric Properties of Wood and Related Issues in Mathematical Description

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Contents

Keywords: *Wood, Elasticity, Piezoelectricity*, *Constitutive behaviour, Averaging of heterogeneous structure*

Historical overview

- 1st publication describing phenomenon of PE (piezo-press) 1880, Curie bros. J & P, experiments on crystals: tourmaline, topaz, quartz, salt (mech. stress → surface charge). Direct PE effect.
- Lippmann G. J. 1881, Converse PE effect (el. charge -> deformation)
- First applications: WW I, Langevin P., Ultrasonic submarine detector
- After: Power sonars, sensitive hydrophones, phono cartridges, microphones, accelerometers, ultrasonic transducers etc.
- Smart & intelligent material structures

Transducers: energy converters

Mechanical energy ⟷ Electrical energy

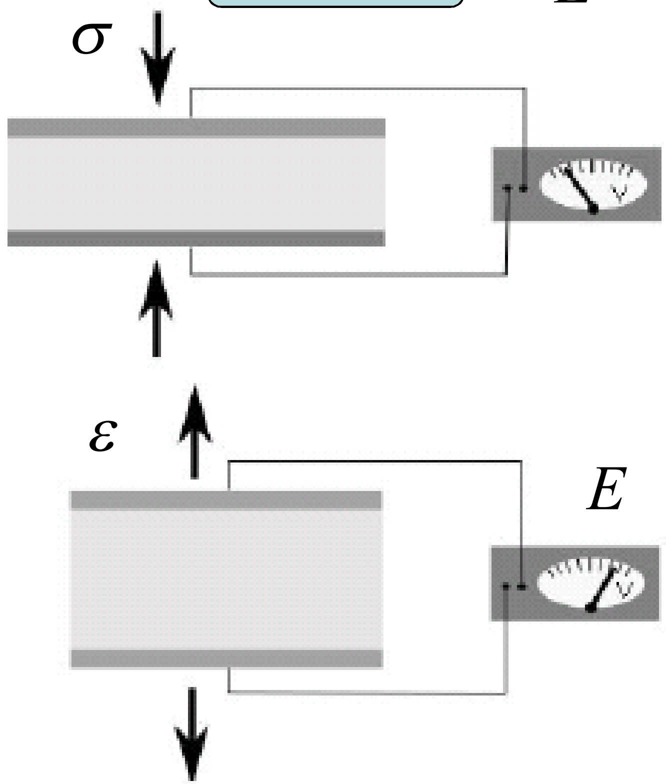
Sensors **Actuators**

Piezo electricity

- direct PE effect
- inverse PE effect

PE system

El. field $E = -\nabla V$ voltage



Direct PE effect

$$Apply \quad \sigma \rightarrow \varepsilon, D$$

El. displacement

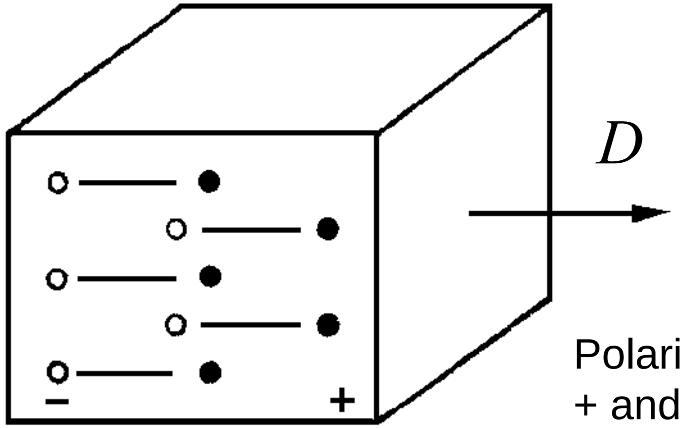
Inverse PE effect

$$Apply \quad E \rightarrow D, \varepsilon$$

PE voltage ⟺ **Mechanical behavior**

Origins of PE

Dielectrics: (nonconductors)



Crystals (mono): space lattice with periodic combination of differently charged ions. El. polarization induced by mechanical stress and deformation and vice versa

PE textures: crystal aggregates formed by oriented crystals with PE properties

Polarized dielectric: polarization-el. displacement + and – el. charges at both ends of the element

Crystal classes: 32

Centro-**S**ymmetric: 11 – FCC, BCC
Non-**C**entro **S**ymm. Exhibit PE behavior

CS: unit cells with point symmetry (center of sym.) w.r.t. all features

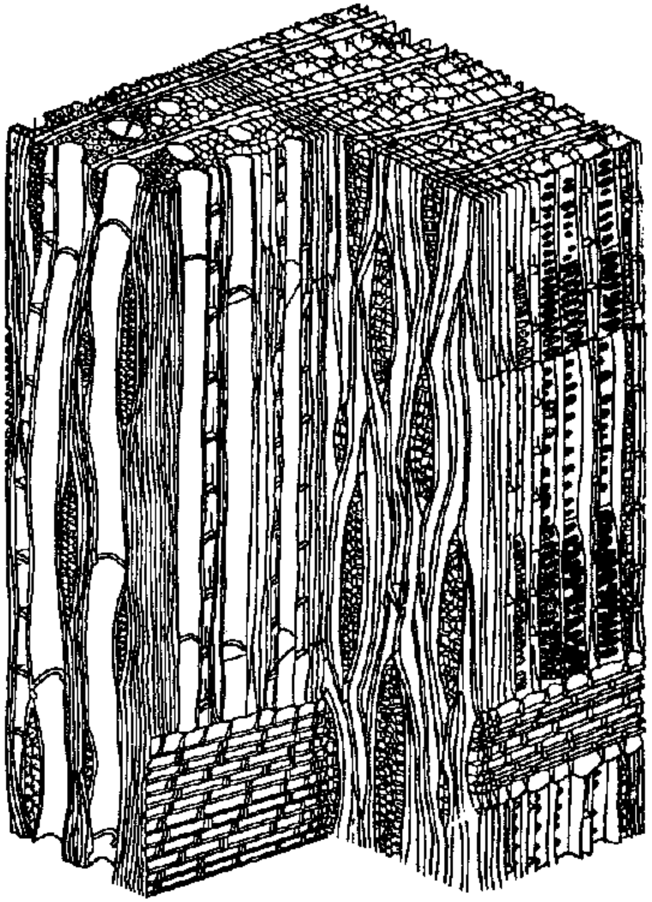
NCS: Ability to form el. dipole & el. dipole moment $\otimes$ $\odot$

P polarization = Resultant of dipole moments

$\boldsymbol{P} = 0$ nonpolar crystal : no PE

$\boldsymbol{P} \neq 0$ polar crystal : PE effect

Relation to wood: cellulose morphology



PE voltage: depends on mechanical deformation of crystalline sub-domains

PE effect: deformation behavior of cellulose (skeleton of wood)

WOOD: natural composite material
Two phase: cellulose & matrix

Crystallization: cellulose crystals in cell wall: stiff micro-fibrils

↓

Stress distribution (loading) in fiber directions

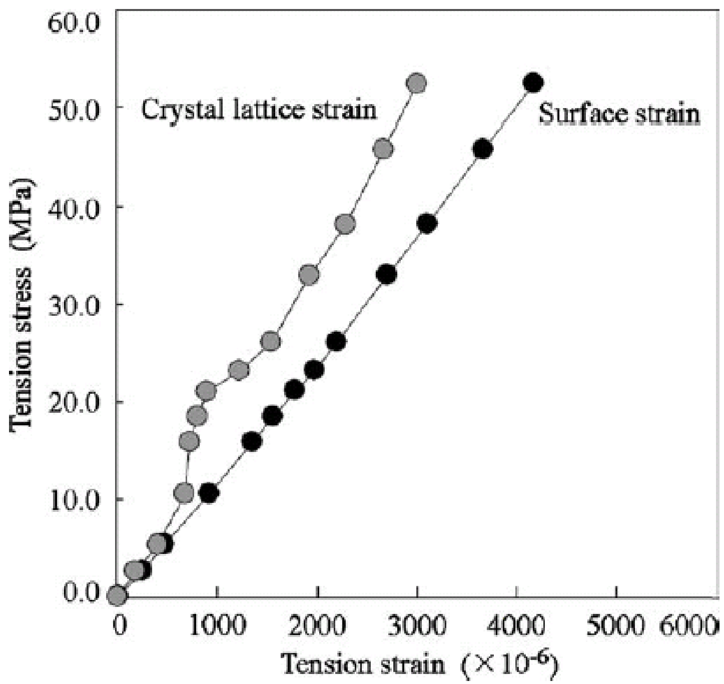
↓

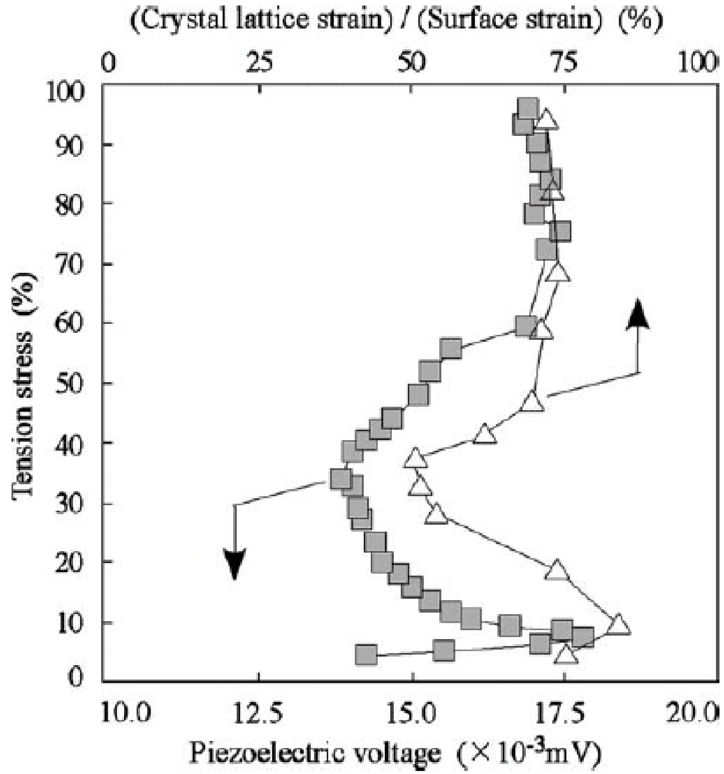
Change in crystal lattice strain

↓

PE effect

Nakai T. et al.: J Wood Sci (1998) 28, 255; (2004) 97;
(2005) 193; (2006) 539;





Wood Sci Technol (2005) 163;

T=20^{o}C, 60% humidity
Density=0.33g/cm^3
Moisture=12%
150X36X7.5mm

Japanese cypress (*Chamaecyparis obtusa*)

Mathematical description

Material body(continuum): $B \subset \mathbb{R}^3$

defined on $B \times [0,\infty), t \in [0,\infty)$

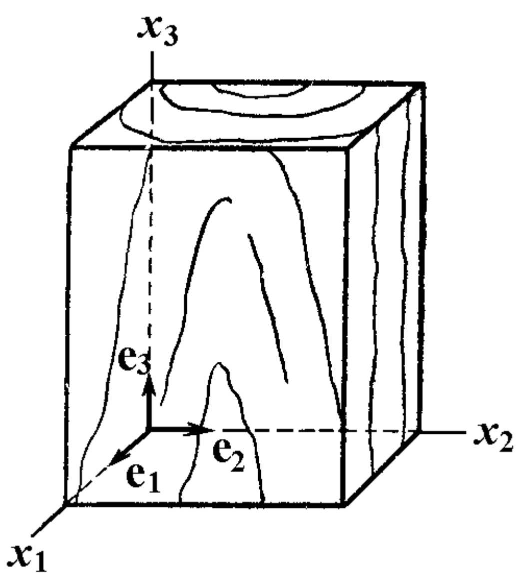
Material point: $P(x) \in B$

Position vector $x : F.C.C.S$

Orthonormal basis $\mathbf{I} = \delta_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$

$$\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}, \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}$$

General frame: arbitrary c.s. orientation w.r.t. bulk element



$$\hat{\mathbf{e}}_i = \mathbb{R}_{ia}\hat{\mathbf{e}}_a, \quad \mathbb{R} \subset SO(3)$$

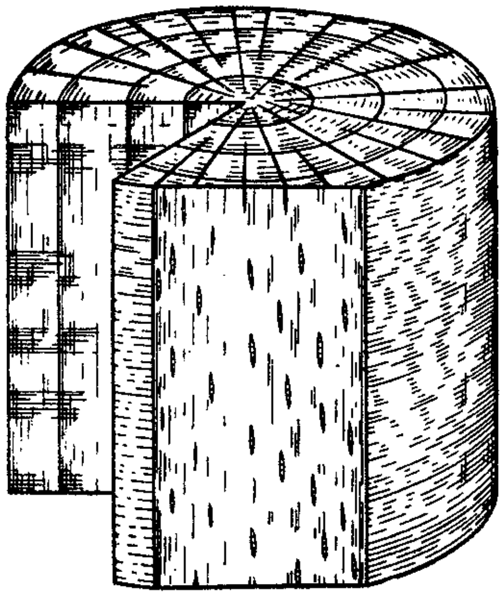
$$\mathbb{R}^{\mathsf{T}}\mathbb{R} = \mathbb{R}\mathbb{R}^{\mathsf{T}} = \mathbf{I}$$

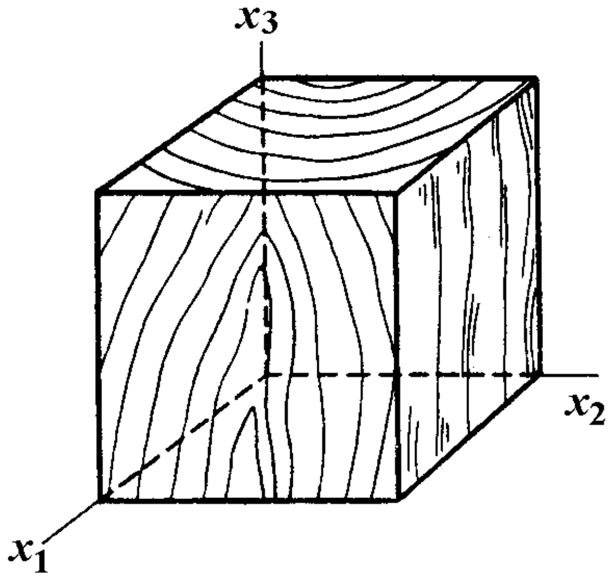
$$\hat{\mathbf{e}}_i = \mathbb{R}_{ia}\hat{\mathbf{e}}_a, \quad \mathbb{R} \subset SO(2) \qquad (x_3 = L)$$

$$\mathbb{R} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Arbitrary coordinate system $\hat{\mathbf{e}}_i = \{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$

Principal coordinate system $\hat{\mathbf{e}}_a = \{\hat{\mathbf{e}}_R, \hat{\mathbf{e}}_T, \hat{\mathbf{e}}_L\}$





$$\hat{\mathbf{e}}_i = \mathbb{R}_{ia} \hat{\mathbf{e}}_a, \quad \mathbb{R} \subset SO(3)$$

General relation between stress and strain

$$\boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\varepsilon}, \quad \sigma_{ij} = C_{ijkl}\varepsilon_{kl} \qquad \boldsymbol{\varepsilon} = \boldsymbol{S} : \boldsymbol{\sigma}, \quad \varepsilon_{ij} = S_{ijkl}\sigma_{kl}$$

Voigt notation

$$\left\{ \begin{array}{c} 11 \to 1,\ 22 \to 2,\ 33 \to 3 \\ 23 = 32 \to 4,\ 31 = 13 \to 5,\ 12 = 21 \to 6 \end{array} \right\}$$

Stress tensor-> vector

$$\left[\sigma_{ij}\right] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_1 & \sigma_6 & \sigma_5 \\ & \sigma_2 & \sigma_4 \\ & & \sigma_3 \end{bmatrix}, \quad \left\{ \begin{array}{c} i, j, k, l = 1 \ldots 3 \\ \alpha, \beta = 1 \ldots 6 \end{array} \right\}$$

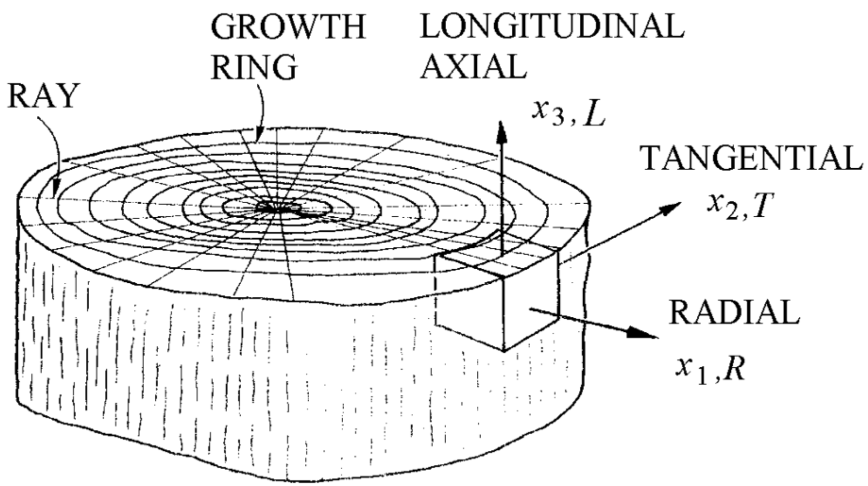
$$\left[\sigma_\alpha\right] = \left\{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\right\}^{\mathrm{T}}, \quad \left[\varepsilon_\alpha\right] = \left\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6\right\}^{\mathrm{T}}$$

$$\boldsymbol{\sigma} = \boldsymbol{C} \cdot \boldsymbol{\varepsilon}, \quad \sigma_\alpha = C_{\alpha\beta}\varepsilon_\beta \qquad \boldsymbol{\varepsilon} = \boldsymbol{S} \cdot \boldsymbol{\sigma}, \quad \varepsilon_\alpha = S_{\alpha\beta}\sigma_\beta$$

Mechanical properties

The mechanical properties of wood are orthotropic: 3 orthogonal planes of symmetry: Radial *(R),* Tangential *(T),* Longitudinal-Axial *(L),* along which the coordinate axes of orthonormal basis are aligned

$$\left\{x, y, z\right\} \equiv \left\{x_1, x_2, x_3\right\} \equiv \left\{R, T, L\right\}$$



$$\boldsymbol{S} = \boldsymbol{S}^{\mathsf{T}} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{21} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{31} & S_{32} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix}$$

$$S_{11} = 1/E_R, S_{12} = -\nu_{RT}/E_T, S_{13} = -\nu_{RL}/E_L,$$

$$S_{21} = -\nu_{TR}/E_T, S_{22} = 1/E_T, S_{23} = -\nu_{TL}/E_L,$$

$$S_{31} = -\nu_{LR}/E_R, S_{32} = -\nu_{LT}/E_T, S_{33} = 1/E_L,$$

$$S_{44} = 1/G_{TL}, S_{55} = 1/G_{LR}, S_{66} = 1/G_{RT}$$

$\{E_R, E_T, E_L\}$…elastic Young's moduli

$\{G_{TL}, G_{LR}, G_{RT}\}$…shear moduli

$\{\nu\}$…the set of Poisson's ratios

General theory of PE

Linear relation between polarization vector and stress (strain)

$$\boldsymbol{D} = D_m \hat{\mathbf{e}}_m = \boldsymbol{d} : \boldsymbol{\sigma} = d_{mij}\sigma_{ij} \hat{\mathbf{e}}_m, \quad i, j, m = 1\ldots3;$$

$$D_m = d_{mij}\sigma_{ij} \qquad \boldsymbol{d} = \begin{bmatrix} d_{111} & d_{122} & d_{133} & d_{123} & d_{131} & d_{112} \\ d_{211} & d_{222} & d_{233} & d_{223} & d_{231} & d_{212} \\ d_{311} & d_{322} & d_{333} & d_{323} & d_{331} & d_{312} \end{bmatrix}$$

$$\left\{ \begin{matrix} 11 \rightarrow 1, 22 \rightarrow 2, 33 \rightarrow 3 \\ 23 = 32 \rightarrow 4, 31 = 13 \rightarrow 5, 12 = 21 \rightarrow 6 \end{matrix} \right\} \qquad \left\{ \begin{matrix} m = 1\ldots3 \\ \alpha = 1\ldots6 \end{matrix} \right\}$$

$$D_m = d_{m\alpha}\sigma_{\alpha} \qquad \boldsymbol{d} = \begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} & d_{15} & d_{16} \\ d_{21} & d_{22} & d_{23} & d_{24} & d_{25} & d_{26} \\ d_{31} & d_{32} & d_{33} & d_{34} & d_{35} & d_{36} \end{bmatrix}$$

WOOD: Piezoelectric properties

Theoretical and Experimental verification: the PE properties for majority of wood species belong to the Schoenflies D2 or 222 international class of orthorhombic non-centro-symmetric systems.

$$\boldsymbol{d} = \begin{bmatrix} 0 & 0 & 0 & d_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & d_{25} & 0 \\ 0 & 0 & 0 & 0 & 0 & d_{36} \end{bmatrix} \qquad \begin{aligned} d_{14} &= 2d_{123} = 2d_{132} \\ d_{25} &= 2d_{231} = 2d_{213} \\ d_{36} &= 2d_{312} = 2d_{321} \end{aligned}$$

$$\left|d_{36}\right| \ll \left(\left|d_{14}\right|, \left|d_{25}\right|\right) \quad \rightarrow \quad d_{36} \approx 0\,?$$

Structure of constitutive equations

$$\boldsymbol{\varepsilon} = \boldsymbol{S} : \boldsymbol{\sigma} + \boldsymbol{E} \cdot \boldsymbol{d}, \quad \varepsilon_{ij} = S_{ijkl}\sigma_{kl} + E_m d_{mij}$$

$$\boldsymbol{D} = \boldsymbol{d} : \boldsymbol{\sigma} + \boldsymbol{\chi} \cdot \boldsymbol{E}, \quad D_m = d_{mij}\sigma_{ij} + \chi_{mk}E_k$$

$$\boldsymbol{\varepsilon} = \boldsymbol{S} \cdot \boldsymbol{\sigma} + \boldsymbol{E} \cdot \boldsymbol{d}, \quad \varepsilon_{\alpha} = S_{\alpha\beta}\sigma_{\beta} + E_m d_{m\alpha}$$

$$\boldsymbol{D} = \boldsymbol{d} \cdot \boldsymbol{\sigma} + \boldsymbol{\chi} \cdot \boldsymbol{E}, \quad D_m = d_{m\beta}\sigma_{\beta} + \chi_{mk}E_k$$

$\sigma \ldots$ *stress vector (tensor)* (N/m^2)

$\varepsilon \ldots$ *strain vector (tensor)* (m/m)

$E \ldots$ *vector of applied el. field* (V/m)

$\chi \ldots$ *permitivity* (F/m)

$d \ldots$ *matrix (tensor) of piezoel. strain constants* (m/V)

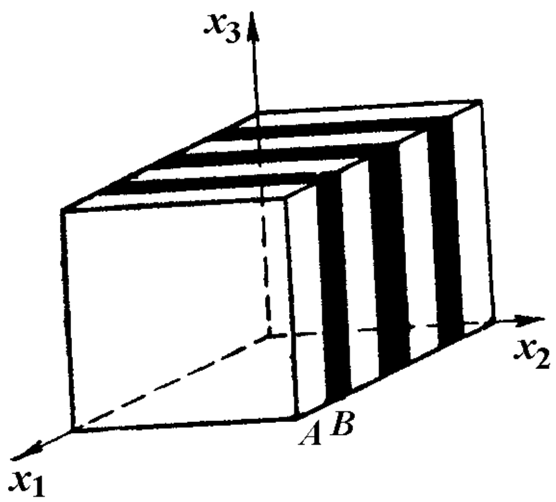
$S \ldots$ *matrix (tensor) of compliance coefficients* (m^2/N)

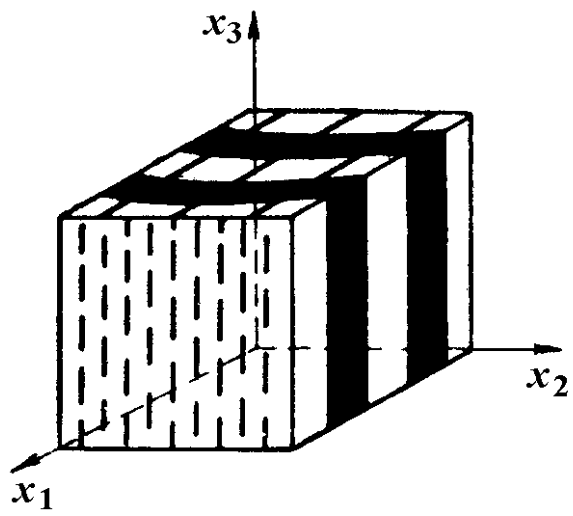
$D \ldots$ *vector of electric polarization displacement* (C/m^2)

Influence of structural heterogeneity on E and PE properties
Micromechanical approach

Problem of averaging of two phase material with different symmetry groups

Anatomical structure of wood: two types of layered PE textures





Prototype problem: **Structural scheme of layered material two phase material: straight annual rings and no xylem rays**

Real problem: **Structural scheme of wood as layered material: small curvature annual rings, xylem rays**

Textures (A, B) - Elastic properties: Symmetry group $\infty:2$

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{13} & 0 & 0 & 0 \\ S_{13} & S_{13} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \tilde{S}_{44} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix}$$

$$\{S_{11}, S_{12}, S_{13}, S_{33}, S_{44}\}_{A,B}$$

$$\tilde{S}_{44} = 2(S_{11} - S_{12})$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \tilde{C}_{44} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{Bmatrix}$$

$$\{C_{11}, C_{12}, C_{13}, C_{33}, C_{44}\}_{A,B}$$

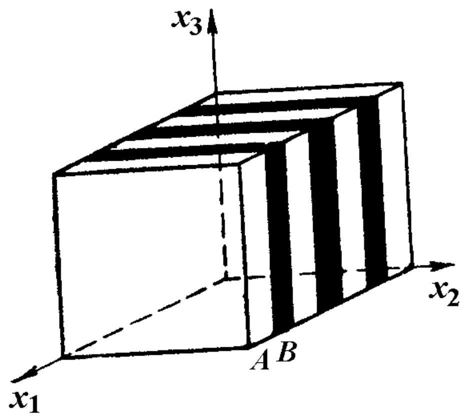
$$\tilde{C}_{44} = (C_{11} - C_{12}) / 2$$

Layered texture A+B: E symmetry group $2:2$

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix}$$

$$\begin{Bmatrix} S_{11}, S_{12}, S_{13}, S_{22}, S_{23} \\ S_{33}, S_{44}, S_{55}, S_{66} \end{Bmatrix}$$

Coupling relations: EL compatibility equations



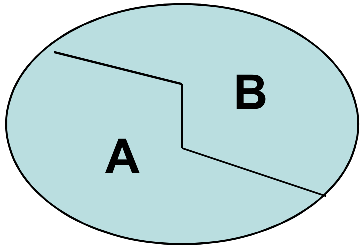
$$\sigma_{11}^{A} = \sigma_{11}^{B} = \sigma_{11}^{A+B} = \langle \sigma_{11} \rangle$$

$$\sigma_{13}^{A} = \sigma_{13}^{B} = \sigma_{13}^{A+B} = \langle \sigma_{13} \rangle$$

$$\sigma_{12}^{A} = \sigma_{12}^{B} = \sigma_{12}^{A+B} = \langle \sigma_{12} \rangle$$

$$\varepsilon_{23}^{A} = \varepsilon_{23}^{B} = \varepsilon_{23}^{A+B} = \langle \varepsilon_{23} \rangle$$

Rule of mixtures



$$V = V_A + V_B$$

$$\xi_A = \frac{V_A}{V}, \quad \xi_B = \frac{V_B}{V}, \quad \xi_A + \xi_B = 1$$

$\xi_A = \xi_B$ Homogeneous texture

$$\left.\begin{aligned} \langle \varepsilon_{ij} \rangle &= \xi_A \varepsilon_{ij}^A + \xi_B \varepsilon_{ij}^B \\ \langle \sigma_{ij} \rangle &= \xi_A \sigma_{ij}^A + \xi_B \sigma_{ij}^B \end{aligned}\right\}$$

Combination of phases **A** & **B** requires more general texture morphology (symmetry group)

$$G_A(\infty:2) \cup G_B(\infty:2) \rightarrow G_{A+B}(2:2)$$

$$\Rightarrow \quad \langle S_{\alpha\beta} \rangle, \langle C_{\alpha\beta} \rangle \in G(2:2)$$

Textures (A, B) - PE properties: Symmetry group $\infty : 2$

$$\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & d_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & d_{25} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{A,B} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix} \qquad d_{25} = -d_{14}$$

Coupling relations: PE compatibility equations

Polarization-el. displacement

$$D_1^A = D_2^A, \quad D_1^B = D_2^B, \quad D_3^A = D_3^B = 0$$

$$\langle D_1 \rangle = \xi_A D_1^A + \xi_B D_1^B$$

$$\langle D_2 \rangle = \xi_A D_2^A + \xi_B D_2^B$$

$$\Rightarrow \quad \langle d_{\alpha\beta} \rangle \in G_{A+B}(2:2)$$

Final remarks

- Wood can be modelled as a two-phase material
- A two phase material structure can take into account the basic morphology of the specimen
- The model can accomodate the effect of the structural inhomogeneity by using structural layer theory, rule of mixtures and first order homogenization - averaging
- The symmetry structure of elastic and PE coupling with the corresponding set of material constants is known
- Can be perceived as a layered texture with changed morphological symmetry with xylem ray texture type of reinforcement in radial direction. The overall resulting symmetry of the texure is (2:2).